随机波动率模型和定价公式

在 Black-Scholes 模型中标的资产的波动率是唯一不能观察的参数,在推导 Black-Scholes 公式时,假设波动率为常数,显然这个假设与实际情况不符合,当实际操作者在运用 Black-Scholes 公式对期权定价时,往往需要随时调整波动率的数值.

相比较而言,假设波动率是随机的,这是一个合理的考虑.

设在鞅测度下,标的资产S_t和波动率σ_t适合以下随机微分方程组:

$$(1)\begin{cases}\dfrac{dS_t}{S_t}=rdt+\sigma_t dW_t\\ \sigma_t=f(Y_t)\\ dY_t=\mu(t,Y_t)dt+\hat{\sigma}(t,Y_t)dZ_t\end{cases}$$

其中$\{W_t: t\geq 0\}$,$\{Z_t: t\geq 0\}$都是标准 Brown 运动,且

$$Cov(dW_t,dZ_t)=\rho dt$$

在随机波动率情况下,欧式期权定价是是一个不完全市场的未定权益定价问题.由于在随机微分方程组中出现两个随机元dW_t和dZ_t,因此单纯用标的资产去对冲,已达不到消除风险的目的.为了对冲由于波动率的随机性所带来的的风险,我们需要在标的资产以外,引入另外一张不同到期日,不同敲定价的期权V_{1t},而组成投资组合

$$\Pi_t=V_t-\Delta_1 S_t-\Delta_2 V_{1t}$$

即希望选取适当的Δ_1和Δ_2,使得Π_t在$[t,t+dt]$时段是无风险的.

由于$V_t=V(S_t,Y_t,t)$,$V_{1t}=V_1(S_t,Y_t,t)$,从而,由微分方程组(1)以及伊藤公式,得到

$$d\Pi_t=dV_t-\Delta_1 dS_t-\Delta_2 dV_{1t}$$

$$=\left[\frac{\partial V}{\partial t}+\frac{1}{2}f^2(Y)S^2\frac{\partial^2 V}{\partial S^2}+\rho f(Y)\hat{\sigma}(t,Y)S\frac{\partial^2 V}{\partial S\partial Y}+\frac{1}{2}\hat{\sigma}^2(t,Y)\frac{\partial^2 V}{\partial Y^2}\right]dt$$

$$-\Delta_2\left[\frac{\partial V_1}{\partial t}+\frac{1}{2}f^2(Y)S^2\frac{\partial^2 V_1}{\partial S^2}+\rho f(Y)\hat{\sigma}(t,Y)S\frac{\partial^2 V_1}{\partial S\partial Y}+\frac{1}{2}\hat{\sigma}^2(t,Y)\frac{\partial^2 V_1}{\partial Y^2}\right]dt$$

$$+\left(\frac{\partial V}{\partial S}-\Delta_2\frac{\partial V_1}{\partial S}-\Delta_1\right)dS_t+\left(\frac{\partial V}{\partial Y}-\Delta_2\frac{\partial V_1}{\partial Y}\right)dY_t \qquad (2)$$

为消去随机性,取Δ_1,Δ_2适合

$$\begin{cases}\dfrac{\partial V}{\partial S}-\Delta_2\dfrac{\partial V_1}{\partial S}-\Delta_1=0\\ \dfrac{\partial V}{\partial Y}-\Delta_2\dfrac{\partial V_1}{\partial Y}=0\end{cases}$$

得到

$$\begin{cases}\Delta_2=\dfrac{\partial V}{\partial Y}\Big/\dfrac{\partial V_1}{\partial Y}\\ \Delta_1=\dfrac{\partial V}{\partial S}-\left(\dfrac{\partial V}{\partial Y}\Big/\dfrac{\partial V_1}{\partial Y}\right)\dfrac{\partial V_1}{\partial S}\end{cases}$$

由Δ_1,Δ_2选取,使得$d\Pi_t$是无风险的,即

$$d\Pi_t=r\Pi_t dt=r(V_t-\Delta_1 S_t-\Delta_2 V_{1t})dt$$

$$=rV_t dt-r\left[\frac{\partial V}{\partial S}-\left(\frac{\partial V}{\partial Y}\Big/\frac{\partial V_1}{\partial Y}\right)\frac{\partial V_1}{\partial S}\right]S_t dt-r\frac{\partial V}{\partial Y}\Big/\frac{\partial V_1}{\partial Y}V_{1t}dt \qquad (3)$$

联立（2）=（3）,并分离V和V_1,得到

$$\left[\frac{\partial V}{\partial t}+\frac{1}{2}f^2(Y)S^2\frac{\partial^2 V}{\partial S^2}+\rho f(Y)\hat{\sigma}(t,Y)S\frac{\partial^2 V}{\partial S\partial Y}\right.$$

$$\left.+\frac{1}{2}\hat{\sigma}^2(t,Y)\frac{\partial^2 V}{\partial Y^2}+rS\frac{\partial V}{\partial S}-rV\right]\Big/\frac{\partial V}{\partial Y}$$

$$=\left[\frac{\partial V_1}{\partial t}+\frac{1}{2}f^2(Y)S^2\frac{\partial^2 V_1}{\partial S^2}+\rho f(Y)\hat{\sigma}(t,Y)S\frac{\partial^2 V_1}{\partial S\partial Y}\right.$$

$$\left.+\frac{1}{2}\hat{\sigma}^2(t,Y)\frac{\partial^2 V_1}{\partial Y^2}+rS\frac{\partial V_1}{\partial S}-rV_1\right]\Big/\frac{\partial V_1}{\partial Y} \qquad (4)$$

由于V和V_1是两个具有不同到期日,不同敲定价的期权,因此等式两端等于一个与期权价V和V_1无关且只依赖于自变量S,Y,t的函数,我们把它记作

$$-\big(\mu(t,Y)-\lambda(t,S,Y)\hat{\sigma}(t,Y)\big) \qquad (5)$$

联立（4）式等式左端与（5）式,导出$V=V(S,Y,t)$适合的偏微分方程

$$\frac{\partial V}{\partial t}+\frac{1}{2}f^2(Y)S^2\frac{\partial^2 V}{\partial S^2}+\rho f(Y)\hat{\sigma}(t,Y)S\frac{\partial^2 V}{\partial S\partial Y}+\frac{1}{2}\hat{\sigma}^2(t,Y)\frac{\partial^2 V}{\partial Y^2}+rS\frac{\partial V}{\partial S}$$
$$+\left(\mu-\lambda\hat{\sigma}(t,Y)\right)\frac{\partial V}{\partial Y}-rV=0 \qquad (6)$$

这里$\lambda=\lambda(t,S,Y)$称为波动率风险的市场价格（market price of volatility）.

注:为了说明λ的金融意义,我们考虑投资组合

$$\Pi_{0t}=V_t-\Delta S_t$$

选Δ,在$[t,t+dt]$时段内消去由标的资产S_t的随机性所带来的风险,即取

$$\Delta=\frac{\partial V}{\partial S}$$

从而我们有

$$\begin{aligned}
&\mathrm{d}\Pi_{0\mathrm{t}}-r\Pi_{0\mathrm{t}}dt\\
&=\left(\frac{\partial V}{\partial t}+\frac{1}{2}f^2(Y)S^2\frac{\partial^2 V}{\partial S^2}+\rho f(Y)\hat{\sigma}(t,Y)S\frac{\partial^2 V}{\partial S\partial Y}+\frac{1}{2}\hat{\sigma}^2(t,Y)\frac{\partial^2 V}{\partial Y^2}\right.\\
&\left.+rS\frac{\partial V}{\partial S}-rV\right)dt+\frac{\partial V}{\partial Y}dY_t\\
&=\hat{\sigma}(t,Y)\frac{\partial V}{\partial Y}(\lambda dt+dZ_t)
\end{aligned}$$

在上面最后一个等式中,我们利用了$V(S,Y,t)$适合方程（6）.上述等式表明由于波动率是随机的,因此在$[t,t+dt]$时段内,对于每一单位波动率风险,存在额外回报λdt,因此人们把λ称为“波动率风险的市场价格”.

人们自然关心如何求出方程（6）带有终值条件为$(S-K)^+$或$(K-S)^+$的欧式期权定价.

对于以下的特殊情形:

（1）若$\sigma_t=f(Y_t)=Y_t^{\frac{1}{2}},\mu(y,Y_t)=\mu Y_t,\hat{\sigma}(t,Y)=\hat{\sigma}Y_t,\rho=0,\lambda=0$,其中$\mu,\hat{\sigma}$为正常数,则欧式看涨期权定价问题为

$$\begin{cases}\frac{\partial V}{\partial t}+\frac{1}{2}YS^2\frac{\partial^2 V}{\partial S^2}+\frac{1}{2}\hat{\sigma}^2Y^2\frac{\partial^2 V}{\partial Y^2}+rS\frac{\partial V}{\partial S}+\mu Y\frac{\partial V}{\partial Y}-rV=0\\ V(S,Y,T)=(S-K)^+\end{cases}$$

Hull 和 White 指出[1],它的解是 Black-Scholes 价格在期权有效期内平均方差概率

分布上的积分值.

（2）若$\sigma_t = f(Y_t) = Y_t^{\frac{1}{2}}, \mu(y, Y_t) = a(\theta - Y_t), \hat{\sigma}(t, Y) = \hat{\sigma}\sqrt{Y_t}, \lambda(t, S, Y_t) = \frac{\lambda}{\hat{\sigma}}\sqrt{Y_t}, |\rho| < 1$,其中$a, \theta, \hat{\sigma}$都是正常数,且满足$a\theta > \frac{1}{2}\hat{\sigma}^2$,$\lambda$是常数,则欧式看涨期权定价问题为

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2} Y S^2 \frac{\partial^2 V}{\partial S^2} + \rho\hat{\sigma} Y S \frac{\partial^2 V}{\partial S \partial Y} + \frac{1}{2}\hat{\sigma}^2 Y \frac{\partial^2 V}{\partial Y^2} + rS\frac{\partial V}{\partial S} + [a(\theta - Y) - \lambda Y]\frac{\partial V}{\partial Y} - rV = 0 \\ V(S, Y, T) = (S - K)^+ \end{cases}$$

S.L.Heston 在文中[2]把$V(S, Y, t)$分解为

$$V(S, Y, t) = SP_1(S, Y, t) - Ke^{-r(T-t)}P_2(S, Y, t)$$

其中$P_j(S, Y, t)(j = 1,2)$在$t = T$时适合

$$P_1(S, Y, T) = P_2(S, Y, T) = H(S - K)$$

$H(x)$是 Heaviside 函数,当$x > 0$ 时,$H(x) = 1$; 当$x < 0$ 时,$H(x) = 0$.

在变换$x = ln\, S$下,$P_j(x, Y, t)(j = 1,2)$适合

$$(7)\begin{cases} \frac{\partial P_j}{\partial t} + \frac{1}{2} Y \frac{\partial^2 P_j}{\partial x^2} + \rho\hat{\sigma} Y \frac{\partial^2 P_j}{\partial x \partial Y} + \frac{1}{2}\hat{\sigma}^2 Y \frac{\partial^2 P_j}{\partial Y^2} + \left(r + \alpha_j Y\right)\frac{\partial P_j}{\partial x} + \left(a\theta - \beta_j Y\right)\frac{\partial P_j}{\partial Y} = 0 & (a) \\ P_j(x, Y, T) = H(x - ln\, K) & (b) \end{cases}$$

其中

$$\alpha_1 = \frac{1}{2}, \alpha_2 = -\frac{1}{2}, \beta_1 = a + \lambda - \rho\hat{\sigma}, \beta_2 = a + \lambda.$$

$P_j(x, Y, t)$的金融意义:若在t时刻,$S_t = e^x$（即$ln\, S_t = x$）,$Y_t = Y$,则在$t = T$时刻,$P_j(x, Y, t)$是期权处于实值状态时的条件期望.

为了求解微分方程组（7）,作者先考虑它的特征函数$f_j(x, Y, t; \varphi)$,这里$f_j(j = 1,2)$适合方程（7-a）,而在$t = T$时,适合终值条件

$$f_j(x, Y, T; \varphi) = e^{i\varphi x} \tag{8}$$

定解问题（7-a）,（8）的解可以表成

$$f_j(x, Y, t; \varphi) = e^{C(T-t;\varphi) + D(T-t;\varphi)Y + i\varphi x}$$

$C(\tau; \varphi), D(\tau; \varphi)$有显示表达式:

$$\begin{cases} C(\tau;\varphi) = r\varphi i\tau + \dfrac{a}{\hat{\sigma}^2}\left\{(b_j - \rho\hat{\sigma}\varphi i + d)\tau - 2ln\left(\dfrac{1-ge^{d\tau}}{1-g}\right)\right\} \\ D(\tau;\varphi) = \dfrac{b_j - \rho\hat{\sigma}\varphi i + d}{\hat{\sigma}^2}\left(\dfrac{1-e^{d\tau}}{1-ge^{d\tau}}\right) \end{cases}$$

其中

$$\begin{cases} g = \dfrac{b_j - \rho\hat{\sigma}\varphi i + d}{b_j - \rho\hat{\sigma}\varphi i - d} \\ d = \sqrt{(\rho\hat{\sigma}\varphi i - b_j)^2 - \hat{\sigma}^2(2\mu_j\varphi i - \varphi^2)} \end{cases}$$

从而由 Fourier 逆变换,定解问题（7）的解$P_j(x,Y,t)(j=1,2)$可表成

$$P_j(x,Y,t) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\varphi ln\,K} f_j(x,Y,t;\varphi)}{i\varphi}\right)d\varphi \tag{9}$$

表达式（9）可以通过数值计算得到.

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